

OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE PREINVEX

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ABSTRACT. In this paper, we established new inequalities of Ostrowski's type for the class of preinvex functions.

1. INTRODUCTION AND PRELIMINARIES

Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the interior of I , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequalities holds:

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right].$$

This result well known in the literature as the Ostrowski's inequality. For recent results and generalizations concerning Ostrowski's inequality see [1, 2] and the references therein.

Definition 1. The function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if the following inequality holds:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The following theorem contains Hadamard's type inequality for M-Lipschitzian functions. (see [10]).

Theorem 1. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be an M-Lipschitzian mapping on I , and $a, b \in I$ with $a < b$. Then we have the inequality:

$$(1.2) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq M \frac{(b-a)}{4}.$$

In [11], and in [12] U.S. Kirmaci proved the following theorems.

Theorem 2. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If the mapping $|f'|$ is convex on $[a, b]$, then we have

$$(1.3) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{8} (|f'(a)| + |f'(b)|)$$

Date: April 6, 2012.

2000 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.

Key words and phrases. Ostrowski type inequalities, preinvex function.

Theorem 3. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and let $p > 1$. If the mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, then we have

$$(1.4) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1} \right)^{\frac{1}{p}} \\ \times \left\{ \left(3|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} + \left(3|f'(b)|^{\frac{p}{p-1}} + |f'(a)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right\}.$$

Theorem 4. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and let $p > 1$. If the mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, then we have

$$(1.5) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{4}{p+1} \right)^{\frac{1}{p}} (|f'(a)| + |f'(b)|).$$

Theorem 5. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and let $p > 1$. If the mapping $|f'|^p$ is convex on $[a, b]$, then

$$(1.6) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \left(\frac{3^{1-\frac{1}{p}}}{8} \right) (b-a) (|f'(a)| + |f'(b)|)$$

In recent years several extentions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [3]. Weir and Mond [4] introduced the concept of preinvex functions and applied it to the establismment of the sufficient optimality conditions and duality in nonlinear programming. Pini [5] introduced the concept of prequasiinvex function as a generalization of invex functions. Later, Mohan and Neogy [9] obtained some properties of generalized preinvex functions. Noor [6] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions.

The aim of this paperis to establish some Ostrowski type inequalities for functions whose derivatives in absolute value are preinvex. Now we recall some notions in invexity analysis which will be used throught the paper (see [7, 8] and references therein)

Let $f : A \rightarrow \mathbb{R}$ and $\eta : A \times A \rightarrow \mathbb{R}$, where A is a nonempty set in \mathbb{R}^n , be continuous functions.

Definition 2. The set $A \subset \mathbb{R}^n$ is said to be invex with respect to $\eta(.,.)$, if for every $x, y \in A$ and $t \in [0, 1]$,

$$x + t\eta(y, x) \in A.$$

The invex set A is also called a η -connected set.

It is obvious that every convex set is invex with respect to $\eta(y, x) = y - x$, but there exist invex sets which are not convex [7].

Definition 3. The function f on the invex set A is said to be preinvex with respect to η if

$$f(x + t\eta(y, x)) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in A, \quad t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

We also need the following assumption regarding the function η which is due to Mohan and Neogy [9]:

Condition C: Let $A \subset \mathbb{R}^n$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$. For any $x, y \in A$ and any $t \in [0, 1]$,

$$\begin{aligned}\eta(y, y + t\eta(x, y)) &= -t\eta(x, y) \\ \eta(x, y + t\eta(x, y)) &= (1 - t)\eta(x, y).\end{aligned}$$

Note that for every $x, y \in A$ and every $t_1, t_2 \in [0, 1]$ from condition C, we have

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y).$$

2. MAIN RESULTS

Lemma 1. *Let $A \subset \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function. If $f' \in L[a, a + \eta(b, a)]$, then the following equality holds:*

$$f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du = \eta(b, a) \left(\int_0^{\frac{x-a}{\eta(b, a)}} t f'(a + t\eta(b, a)) dt + \int_{\frac{x-a}{\eta(b, a)}}^1 (t-1) f'(a + t\eta(b, a)) dt \right)$$

for all $x \in [a, a + \eta(b, a)]$.

Since $A \subset \mathbb{R}$ is an invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$, for all $t \in [0, 1]$ we have $a + t\eta(b, a) \in A$. A simple proof of equality can be given by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader.

The following result may be stated:

Theorem 6. *Let $A \supset [0, \infty)$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $0 \leq a < a + \eta(b, a) < \infty$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L[a, a + \eta(b, a)]$. If $|f'|$ is preinvex function on A then the following inequality holds:*

$$\begin{aligned}(2.1) \quad & \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \\ & \leq \eta(b, a) \left\{ \begin{aligned} & \left[\frac{1}{2} \left(\frac{x-a}{\eta(b, a)} \right)^2 - \frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 + \frac{1}{3} \left(\frac{a+\eta(b, a)-x}{\eta(b, a)} \right)^3 \right] |f'(a)| + \\ & \left[\frac{1}{6} - \frac{1}{2} \left(\frac{x-a}{\eta(b, a)} \right)^2 + \frac{2}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 \right] |f'(b)| \end{aligned} \right\} \end{aligned}$$

Proof. By lemma 1 and since $|f'|$ is preinvex, we have

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \\
& \leq \eta(b, a) \left\{ \int_0^{\frac{x-a}{\eta(b, a)}} t |f'(a + t\eta(b, a))| dt + \int_{\frac{x-a}{\eta(b, a)}}^1 (1-t) |f'(a + t\eta(b, a))| dt \right\} \\
& \leq \eta(b, a) \left\{ \int_0^{\frac{x-a}{\eta(b, a)}} t [(1-t) |f'(a)| + t |f'(b)|] dt + \int_{\frac{x-a}{\eta(b, a)}}^1 (1-t) [(1-t) |f'(a)| + t |f'(b)|] dt \right\} \\
& \leq \eta(b, a) \left\{ \begin{aligned} & \left[\frac{1}{2} \left(\frac{x-a}{\eta(b, a)} \right)^2 - \frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 + \frac{1}{3} \left(\frac{a+\eta(b, a)-x}{\eta(b, a)} \right)^3 \right] |f'(a)| + \\ & \left[\frac{1}{6} - \frac{1}{2} \left(\frac{x-a}{\eta(b, a)} \right)^2 + \frac{2}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 \right] |f'(b)| \end{aligned} \right\}
\end{aligned}$$

where we have used the fact that

$$\int_0^{\frac{x-a}{\eta(b, a)}} t(1-t) dt + \int_{\frac{x-a}{\eta(b, a)}}^1 (1-t)^2 dt = \frac{1}{2} \left(\frac{x-a}{\eta(b, a)} \right)^2 - \frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 + \frac{1}{3} \left(\frac{a+\eta(b, a)-x}{\eta(b, a)} \right)^3$$

and

$$\int_0^{\frac{x-a}{\eta(b, a)}} t^2 dt + \int_{\frac{x-a}{\eta(b, a)}}^1 t(1-t) dt = \frac{1}{6} - \frac{1}{2} \left(\frac{x-a}{\eta(b, a)} \right)^2 + \frac{2}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3.$$

□

Remark 1. Suppose that all the assumptions of Theorem 6 are satisfied. Then

(a) If we choose $\eta(b, a) = b - a$ and $x = \frac{2a+\eta(b, a)}{2}$ we obtain

$$(2.2) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|)$$

which is the inequality (1.3).

(b) If in addition we choose $|f'(x)| \leq M$, $M > 0$ in (a), then we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq M \frac{(b-a)}{4}$$

which is the inequality (1.2).

(c) if the mapping η satisfies condition C then for every $t \in [0, 1]$, it yields

$$\begin{aligned}
(2.3) \quad f(a + t\eta(b, a)) &= f(a + \eta(b, a) + (1-t)\eta(a, a + \eta(b, a))) \\
&\leq tf(a + \eta(b, a)) + (1-t)f(a).
\end{aligned}$$

Using inequality (2.3) for $|f'|$ in the proof of Theorem 6, (2.1) become the following inequality:

$$\begin{aligned} & \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \\ & \leq \eta(b, a) \left\{ \begin{aligned} & \left[\frac{1}{2} \left(\frac{x-a}{\eta(b, a)} \right)^2 - \frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 + \frac{1}{3} \left(\frac{a+\eta(b, a)-x}{\eta(b, a)} \right)^3 \right] |f'(a)| + \\ & \left[\frac{1}{6} - \frac{1}{2} \left(\frac{x-a}{\eta(b, a)} \right)^2 + \frac{2}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 \right] |f'(a + \eta(b, a))| \end{aligned} \right\} \end{aligned}$$

Theorem 7. Let $A \supset [0, \infty)$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $0 \leq a < a + \eta(b, a) < \infty$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L[a, a + \eta(b, a)]$ and η satisfies condition C. If $|f'|^q$ is preinvex function on $[a, a + \eta(b, a)]$ for some fixed $q > 1$ then for each $x \in [a, a + \eta(b, a)]$ the following inequality holds:

$$\begin{aligned} (2.4) \quad & \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \\ & \times \left\{ \frac{(x-a)^2}{\eta(b, a)} \left(\frac{|f'(a)|^q + |f'(x)|^q}{2} \right)^{\frac{1}{q}} + \frac{(a+\eta(b, a)-x)^2}{\eta(b, a)} \left(\frac{|f'(a+\eta(b, a))|^q + |f'(x)|^q}{2} \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We first note that if $|f'|^q$ is a preinvex function on $[a, a + \eta(b, a)]$ and the mapping η satisfies condition C then for every $t \in [0, 1]$, it yields the inequality (2.3) and similarly

$$\begin{aligned} (2.5) \quad & |f'(a + (1-t)\eta(b, a))|^q = |f'(a + \eta(b, a) + t\eta(a, a + \eta(b, a)))|^q \\ & \leq (1-t) |f'(a + \eta(b, a))|^q + t |f'(a)|^q. \end{aligned}$$

By adding these inequalities we have

$$(2.6) \quad |f'(a + t\eta(b, a))|^q + |f'(a + (1-t)\eta(b, a))|^q \leq |f'(a)|^q + |f'(a + \eta(b, a))|^q$$

Then integrating the inequality (2.6) with respect to t over $[0, 1]$, we obtain

$$(2.7) \quad \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} |f'(t)|^q dt \leq \frac{|f'(a)|^q + |f'(a + \eta(b, a))|^q}{2}$$

From lemma 1 and using Hölder inequality, we have

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \\
& \leq \eta(b,a) \left(\int_0^{\frac{x-a}{\eta(b,a)}} t^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} |f'(a+t\eta(b,a))|^q dt \right)^{\frac{1}{q}} \\
& \quad + \eta(b,a) \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 |f'(a+t\eta(b,a))|^q dt \right)^{\frac{1}{q}} \\
& \leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \frac{(x-a)^2}{\eta(b,a)} \left(\frac{|f'(a)|^q + |f'(x)|^q}{2} \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \frac{(a+\eta(b,a)-x)^2}{\eta(b,a)} \left(\frac{|f'(a+\eta(b,a))|^q + |f'(x)|^q}{2} \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

where we use the fact that

$$\int_0^{\frac{x-a}{\eta(b,a)}} t^p dt = \frac{1}{p+1} \left(\frac{x-a}{\eta(b,a)} \right)^{p+1}, \quad \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^p dt = \frac{1}{p+1} \left(\frac{a+\eta(b,a)-x}{\eta(b,a)} \right)^{p+1},$$

and by (2.7) we get

$$\begin{aligned}
& \int_0^{\frac{x-a}{\eta(b,a)}} |f'(a+t\eta(b,a))|^q dt \leq \frac{x-a}{\eta(b,a)} \left(\frac{|f'(a)|^q + |f'(x)|^q}{2} \right), \\
& \int_{\frac{x-a}{\eta(b,a)}}^1 |f'(a+t\eta(b,a))|^q dt \leq \frac{a+\eta(b,a)-x}{\eta(b,a)} \left(\frac{|f'(a+\eta(b,a))|^q + |f'(x)|^q}{2} \right).
\end{aligned}$$

□

Corollary 1. Suppose that all the assumptions of Theorem 7 are satisfied. If we choose $|f'(x)| \leq M$, $M > 0$, for each $x \in [a, a+\eta(b,a)]$, then we have

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} M \left(\frac{(x-a)^2 + (a+\eta(b,a)-x)^2}{\eta(b,a)} \right)$$

Corollary 2. Suppose that all the assumptions of Theorem 7 are satisfied. we let $x = \frac{2a+\eta(b,a)}{2}$, then Since $|f'|^q$ is a preinvex function on $[a, a+\eta(b,a)]$ by inequality

(2.3) for $t = \frac{1}{2}$ we have

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \times \left\{ \frac{\eta(b,a)}{4} \left(\frac{3|f'(a)|^q + |f'(a+\eta(b,a))|^q}{4} \right)^{\frac{1}{q}} + \frac{\eta(b,a)}{4} \left(\frac{3|f'(a+\eta(b,a))|^q + |f'(a)|^q}{4} \right)^{\frac{1}{q}} \right\}$$

Remark 2. In Corollary 2 we let $\eta(b, a) = b - a$, then we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{\frac{1}{p}} \left\{ (3|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} + (3|f'(b)|^q + |f'(a)|^q)^{\frac{1}{q}} \right\} \end{aligned}$$

which is the inequality (1.4). Let $a_1 = 3|f'(a)|^q$, $b_1 = |f'(b)|^q$, $a_2 = 3|f'(b)|^q$, $b_2 = |f'(a)|^q$. Here $0 < \frac{1}{q} < 1$, for $q > 1$. Using the fact that

$$(2.8) \quad \sum_{k=1}^n (a_k + b_k)^s \leq \sum_{k=1}^n a_k^s + \sum_{k=1}^n b_k^s$$

for $(0 \leq s < 1)$, $a_1, a_2, \dots, a_n \geq 0$, $b_1, b_2, \dots, b_n \geq 0$, we obtain

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{\frac{1}{p}} (|f'(a)| + |f'(b)|)$$

which is the inequality (1.5).

Theorem 8. Let $A \supset [0, \infty)$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $0 \leq a < a + \eta(b, a) < \infty$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L[a, a + \eta(b, a)]$. If $|f'|^q$ is preinvex function on $[a, a + \eta(b, a)]$ for some fixed $q \geq 1$ then for each $x \in [a, a + \eta(b, a)]$ the following inequality holds:

$$\begin{aligned} & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \\ & \leq \eta(b,a) \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{x-a}{\eta(b,a)}\right)^{2(1-\frac{1}{q})} \left[\frac{(x-a)^2(3\eta(b,a)-2x+2a)}{6\eta^3(b,a)} |f'(a)|^q + \frac{1}{3} \left(\frac{x-a}{\eta(b,a)}\right)^3 |f'(b)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{2(1-\frac{1}{q})} \left[\frac{1}{3} \left(\frac{x-a}{\eta(b,a)}\right)^3 |f'(a)|^q + \left(\frac{1}{6} + \frac{(x-a)^2(2x-3\eta(b,a)-2a)}{6\eta^3(b,a)}\right) |f'(b)|^q \right]^{\frac{1}{q}} \right\} \end{aligned}$$

Proof. By Lemma 1 and inequality (2.3), and using the well known power mean inequality, we have

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \\
& \leq \eta(b, a) \left(\int_0^{\frac{x-a}{\eta(b, a)}} t dt \right)^{1-\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b, a)}} t |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\
& \quad + \eta(b, a) \left(\int_{\frac{x-a}{\eta(b, a)}}^1 (1-t) dt \right)^{1-\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b, a)}}^1 (1-t) |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\
& \leq \eta(b, a) \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left\{ \begin{aligned} & \left(\frac{x-a}{\eta(b, a)} \right)^{2(1-\frac{1}{q})} \left[\frac{(x-a)^2 (3\eta(b, a) - 2x + 2a)}{6\eta^3(b, a)} |f'(a)|^q \right. \\ & \quad \left. + \frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 |f'(b)|^q \right]^{\frac{1}{q}} \\ & + \left(\frac{a+\eta(b, a)-x}{\eta(b, a)} \right)^{2(1-\frac{1}{q})} \left[\frac{\frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 |f'(a)|^q}{+ \left(\frac{1}{6} + \frac{(x-a)^2 (2x-3\eta(b, a)-2a)}{6\eta^3(b, a)} \right) |f'(b)|^q} \right]^{\frac{1}{q}} \end{aligned} \right\}
\end{aligned}$$

where we use the fact that

$$\begin{aligned}
& \int_0^{\frac{x-a}{\eta(b, a)}} t dt = \frac{1}{2} \left(\frac{x-a}{\eta(b, a)} \right)^2, \\
& \int_0^{\frac{x-a}{\eta(b, a)}} t |f'(a + t\eta(b, a))|^q dt \\
& \leq \frac{(x-a)^2 (3\eta(b, a) - 2x + 2a)}{6\eta^3(b, a)} |f'(a)|^q + \frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 |f'(b)|^q, \\
& \int_{\frac{x-a}{\eta(b, a)}}^1 (1-t) dt = \frac{1}{2} \left(\frac{a+\eta(b, a)-x}{\eta(b, a)} \right)^2, \\
& \int_{\frac{x-a}{\eta(b, a)}}^1 (1-t) |f'(a + t\eta(b, a))|^q dt \\
& \leq \frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 |f'(a)|^q + \left(\frac{1}{6} + \frac{(x-a)^2 (2x-3\eta(b, a)-2a)}{6\eta^3(b, a)} \right) |f'(b)|^q.
\end{aligned}$$

The proof is completed. \square

Corollary 3. Suppose that all the assumptions of Theorem 8 are satisfied. we let $x = \frac{2a+\eta(b,a)}{2}$, then by the inequality (2.8) we have

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \left(\frac{3^{1-\frac{1}{q}}}{8}\right) \eta(b,a) (|f'(a)| + |f'(b)|).$$

Remark 3. Suppose that all the assumptions of Theorem 8 are satisfied.

(a) In Corollary 3 we let $\eta(b,a) = b-a$, then we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \left(\frac{3^{1-\frac{1}{q}}}{8}\right) (b-a) (|f'(a)| + |f'(b)|)$$

which is the inequality (1.6).

(b) if the mapping η satisfies condition C then using inequality (2.3) for $|f'|^q$ in the proof of Theorem 8, we get

$$\begin{aligned} & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \\ & \leq \eta(b,a) \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ \begin{aligned} & \left(\frac{x-a}{\eta(b,a)}\right)^{2(1-\frac{1}{q})} \left[\frac{(x-a)^2(3\eta(b,a)-2x+2a)}{6\eta^3(b,a)} |f'(a)|^q \right. \\ & \quad \left. + \frac{1}{3} \left(\frac{x-a}{\eta(b,a)}\right)^3 |f'(a+\eta(b,a))|^q \right]^{\frac{1}{q}} \\ & + \left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{2(1-\frac{1}{q})} \left[\frac{1}{3} \left(\frac{x-a}{\eta(b,a)}\right)^3 |f'(a)|^q \right. \\ & \quad \left. + \left(\frac{1}{6} + \frac{(x-a)^2(2x-3\eta(b,a)-2a)}{6\eta^3(b,a)}\right) |f'(a+\eta(b,a))|^q \right]^{\frac{1}{q}} \end{aligned} \right\} \end{aligned}$$

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